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THE EFFECT OF TRANSVERSE SHEAR STRESSES ON THE YIELD SURFACE FOR THIN SHELLS

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Abstract—Several approximate yield surfaces for a thin shell have been examined in the general case where transverse shear stresses are not negligible. It is found that if a simple modification is made to existing approximations, the resulting bounds on the limit load are nearly identical to those obtained from a simple shell theory where transverse shear is neglected. A new expression accurate to plus or minus 2 per cent is also derived.

NOTATION

σ_0	yield stress of material in simple tension
T	thickness of shell
n_1, n_2, n_{12}; q_1, q_2	dimensionless stress resultants acting on a shell element ($n = N/\sigma_0 T$, $q = Q/\sigma_0 T$)
m_1, m_2, m_{12}	dimensionless bending moments acting on a shell element ($m = 4M/\sigma_0 T^2$)
Q_t	$n_1^2+n_2^2-n_1n_2+3n_{12}^2$
Q_m	$m_1^2+m_2^2-m_1m_2+3m_{12}^2$
Q_{tm}	$n_1m_1-\frac{1}{2}n_1m_2-\frac{1}{2}n_2m_1+n_2m_2+3n_{12}m_{12}$
Q_q	$3q_1^2+3q_2^2$
e_1, e_2, e_{12}; e_{13}, e_{23}	strain rate components
k_1, k_2, k_{12}	curvature rate components
P_e	$e_1^2+e_2^2+e_1e_2+\frac{1}{4}e_{13}^2+\frac{1}{4}e_{23}^2+\frac{1}{4}e_{12}^2$
P_{ek}	$e_1k_1+e_2k_2+\frac{1}{2}e_1k_2+\frac{1}{2}e_2k_1+\frac{1}{4}e_{12}k_{12}$
P_k	$k_1^2+k_2^2+k_1k_2+\frac{1}{4}k_{12}^2$
P_q	$\frac{1}{4}e_{23}^2+\frac{1}{4}e_{13}^2$

The quadratic forms Q_t, Q_m, Q_q, P_e, P_k, P_q are positive definite. It is assumed that the shell is thin so that the differences between N_{12} and N_{21} and M_{12} and M_{21} can be ignored (see [1]).

1. INTRODUCTION

IN A previous paper [1] a comparison was made of various approximate yield surfaces for thin shells. The basis of the work was the investigation of plastic shells by A. A. Ilyushin [2] in which the usual assumptions of plane cross-sections remaining plane and normal and the neglect of transverse shear were made. The material was assumed to be isotropic, to obey the von Mises yield criterion and the shell was assumed to yield throughout its thickness. A yield surface dependent only on stress resultants was derived and more specifically it was shown that the yield surface must be of the form $F(Q_t, Q_m, Q_{tm}) = 0$. Having obtained a parametric form of the yield surface in terms of two parameters λ and μ a comparison was then made in [1] of several approximations which are in frequent use, and bounds obtained on the resulting limit load. Throughout all this the effect of transverse shear was neglected entirely but a modification was suggested if it was thought that shear stress resultants could be large. It was proposed that Q_t be augmented by Q_q for whichever approximation was used and that Ilyushin's results would require no other modification.

This suggestion was objected to [3] and in reply it was conceded that in most cases simply adding Q_q to Q_t was not strictly accurate. The problem therefore remains open and it is the purpose of this paper to investigate the effect of transverse shear more thoroughly. Such an investigation is necessary because in certain circumstances the shear effect is pronounced and has to be accounted for, even if only approximately. An obvious example is a circular plate under concentrated loading. Sawczuk and Duszek [4] have given a theoretical analysis of this case for a uniformly distributed pressure loading, although linearization of the yield surface was necessary to obtain an analytical solution. Ellyin and Deloin [5] have considered the effect of shear on the yielding of arches under a line load and have shown that for certain geometries this effect is appreciable. Several remarks on the general topic of shear were made in the reply in [3] but for the sake of completeness the conclusions will be restated here.

First of all it must be borne in mind that the present investigation is limited to thin shells only, and therefore employs the concept of the generalized stress resultant. Associated with these stress resultants are generalized strain rates such that the sum of products of stress resultants and strain rates gives the work rate per unit area. Having obtained such a set of stress resultants and strain rates, the lower and upper bound theorems of limit analysis can be employed.

Now if a "first order" shell theory is used, that is, we use direct stress resultants and bending moments, we *must* impose the kinematic condition that plane sections normal to the mid-surface remain plane, i.e. the strain rate distribution through the shell thickness is linear. They do not have to remain normal if we take account of the transverse shear stress resultant. The usual shell limit analysis theorems then follow without any trouble if work done at any plastic hinges is allowed for, (where discontinuities such as of slope or mid surface occur). The appropriate generalization of Ilyushin's results, in parametric form, has been given by Shapiro [6]. If this kinematic restraint is not imposed then the stress resultants used are strictly of no relevance and the upper and lower bound theorems of limit analysis, with their associated convexity properties, do not necessarily apply. A good discussion on these topics has been given by Heyman [7]. This point has not always been recognized sufficiently in the literature, where such ideas as non-linear strain rate distribution, linear stress distribution, and yield averaging have been proposed. These suggestions should be viewed with suspicion as they have no sound theoretical justification.

Having been forced to conclude that the Shapiro extension is the only possible one for a shell theory, it is necessary to ask whether shell theory is always adequate to account for the shear effect. The actual situation is of course complicated by the fact that the collapse mechanism may be highly localized and irregular, and in general, since the shear stress (and hence strain rate) must vanish at a free surface, the shear strain rate cannot be linear through the thickness. Nevertheless, Hodge [8] considered beam bending and he showed that a stress resultant (i.e. shell) theory which included shear gave good results except for very extreme geometries where the length of the beam was less than about twice the depth. Both Ellyin and Hodge assumed linear transverse strain rates and therefore used a particular form, restricted to two or three dimensions, of the more general eight dimensional yield surface to be analyzed in this paper.

In conclusion then we can summarize the above as follows. The effect of shear on the yielding of structures cannot always be neglected and some attempt must be made to account for it. The Shapiro yield surface is the only possible one for a shell theory because of certain kinematic constraints inherent in shell theory. Although not perfect it gives reliable results for most cases of beam bending, and will extend the range of applicability of shell limit analysis to many structures where a "no shear" theory is seriously in error. There will however remain cases where shell theory is inadequate and a full three-dimensional analysis must be done. Finally, in most structural problems it is tacitly assumed that shear is negligible. This is usually justified but the analysis to follow will then give some estimate of the error involved.

2. THE PARAMETRICAL REPRESENTATION OF SHAPIRO'S YIELD SURFACE

The following definitions and results are taken from Ref. 6 with any necessary changes in notation. z is thickness dimension and σ_i are stress components.

$$\sigma_0 T n_1 = \int_{-T/2}^{T/2} \sigma_1 \, dz = \frac{\sigma_0}{\sqrt{3}}[(2e_1+e_2)I_1+(2k_1+k_2)I_2]$$

$$\sigma_0 T n_2 = \int_{-T/2}^{T/2} \sigma_2 \, dz = \frac{\sigma_0}{\sqrt{3}}[(e_1+2e_2)I_1+(k_1+2k_2)I_2]$$

$$\sigma_0 T n_{12} = \int_{-T/2}^{T/2} \sigma_{12} \, dz = \frac{\sigma_0}{2\sqrt{3}}[e_{12}I_1+k_{12}I_2]$$

$$\sigma_0 T q_1 = \int_{-T/2}^{T/2} \sigma_{13} \, dz = \frac{\sigma_0 e_{13}}{2\sqrt{3}} I_1$$

$$\sigma_0 T q_2 = \int_{-T/2}^{T/2} \sigma_{23} \, dz = \frac{\sigma_0 e_{23}}{2\sqrt{3}} I_1$$

$$\tfrac{1}{4}\sigma_0 T^2 m_1 = \int_{-T/2}^{T/2} \sigma_1 z \, dz = \frac{\sigma_0}{\sqrt{3}}[(2e_1+e_2)I_2+(2k_1+k_2)I_3]$$

$$\tfrac{1}{4}\sigma_0 T^2 m_2 = \int_{-T/2}^{T/2} \sigma_2 z \, dz = \frac{\sigma_0}{\sqrt{3}}[(e_1+2e_2)I_2+(k_1+2k_2)I_3]$$

$$\tfrac{1}{4}\sigma_0 T^2 m_{12} = \int_{-T/2}^{T/2} \sigma_{12} z \, dz = \frac{\sigma_0}{2\sqrt{3}}[e_{12}I_2+k_{12}I_3]$$

where

$$P = \sqrt{(P_e+2zP_{ek}+z^2P_k)}$$

and

$$I_1 = \int_{-T/2}^{T/2} \frac{dz}{P}, \qquad I_2 = \int_{-T/2}^{T/2} \frac{z\,dz}{P}, \qquad I_3 = \int_{-T/2}^{T/2} \frac{z^2\,dz}{P}.$$

From the above definitions it can be seen that

$$\left.\begin{aligned} T^2Q_t &= I_1^2(P_e-P_q)+2I_1I_2P_{ek}+I_2^2P_k \\ \tfrac{1}{4}T^3Q_{tm} &= I_1I_2(P_e-P_q)+(I_1I_3+I_2^2)P_{ek}+I_2I_3P_k \\ \tfrac{1}{16}T^4Q_m &= I_2^2(P_e-P_q)+2I_2I_3P_{ek}+I_3^2P_k \\ T^2Q_q &= I_1^2P_q \end{aligned}\right\} . \tag{1}$$

Since P_e, P_{ek}, P_k define I_1, I_2 and I_3 the yield surface is of the form $F(Q_t, Q_q, Q_{tm}, Q_m) = 0$. Consider first the interaction surface for $Q_t = 0$, i.e. $n_1 = n_2 = n_{12} = 0$. This gives

$$e_1k_1 = e_2/k_2 = e_{12}/k_{12} = -I_2/I_1 = c \quad \text{say} \tag{2}$$

therefore $P_e-P_q = c^2P_k$ and $P_{ek} = cP_k$. If $P_k = 0$ then $Q_t = Q_m = Q_{tm} = 0$, $Q_q = 1$. If $P_k > 0$ then from (2) we get $cI_1+I_2 = 0$ or

$$\int_{-T/2}^{T/2} \frac{c+z}{\sqrt{[(c+z)^2+P_q/P_k]}}\,\mathrm{d}z = 0.$$

It follows that $c = 0$ and hence $P_e = P_q$ and $I_2 = 0$. After simple algebra we obtain

$$Q_q = \frac{\mu_1}{4}\left(\log_e\left(\frac{\sqrt{(1+\mu_1)}+1}{\sqrt{(1+\mu_1)}-1}\right)\right)^2, \; Q_m = \left(\sqrt{(1+\mu_1)}-\frac{\mu_1}{2}\log_e\left(\frac{\sqrt{(1+\mu_1)}+1}{\sqrt{(1+\mu_1)}-1}\right)\right)^2 \tag{3}$$

where $0 \leq \mu_1 < \infty$. $Q_q = 1$, $Q_m = 0$ and $Q_m = 1$, $Q_q = 0$ are obtained as limiting cases.

In examining $Q_m = 0$, we now get $k_1/e_1 = k_2/e_2 = k_{12}/e_{12} = -I_2/I_3 = c$. Therefore $P_k = c^2(P_e-P_q)$ and $P_{ek} = c(P_e-P_q)$. Substituting in $I_2+cI_3 = 0$ leads to

$$\frac{1}{\sqrt{(P_e-P_q)}}\int_{-T/2}^{T/2} \frac{z+cz^2}{\sqrt{[(1+cz)^2+P_q/(P_e-P_q)]}}\,\mathrm{d}z = 0.$$

If $P_e = P_q$ we have $P_k = P_{ek} = 0$ and thus $Q_t = 0$, $Q_q = 1$. If $P_q = 0$ then $|c| \leq 2/T$ but is otherwise unrestricted. After simple algebra we get $Q_q = 0$, $Q_t = 1$. It can also be shown by examining the above integral that if $P_q > 0$, $P_e > P_q$ then we must have $c = 0$ and thus $Q_t+Q_q = 1$. Therefore when $Q_m = 0$ the equation of the yield surface is

$$Q_t+Q_q = 1. \tag{4}$$

We will now obtain the general parametric form of the yield surface. We will assume $P_k > 0$ for if $P_k = 0$ then $Q_t+Q_q = 1$ as is seen from (1). Without loss of generality we will also assume $P_{ek} \geq 0$ since sign changes in P_k, keeping P_e fixed, do not alter Q_t, Q_m or Q_q as is verified by examining (1). Q_{tm} merely changes sign.

3. GENERAL ANALYSIS

We will make the following definitions:

$$\lambda = \sqrt{\frac{P_e+\frac{1}{4}T^2P_k-TP_{ek}}{P_e+\frac{1}{4}T^2P_k+TP_{ek}}}, \qquad \mu = \sqrt{\frac{P_e-P_{ek}^2/P_k}{P_e+\frac{1}{4}T^2P_k+TP_{ek}}}, \qquad G = P_e+\tfrac{1}{4}T^2P_k+TP_{ek}. \tag{5}$$

Therefore

$$\lambda^2-\mu^2 = \frac{P_k}{G}\left[\frac{T}{2}-\frac{P_{ek}}{P_k}\right]^2 \quad \text{and} \quad 1-\mu^2 = \frac{P_k}{G}\left[\frac{T}{2}+\frac{P_{ek}}{P_k}\right]^2$$

and thus

$$\frac{T}{2}-\frac{P_{ek}}{P_k} = \pm\sqrt{\left(\frac{G}{P_k}(\lambda^2-\mu^2)\right)}, \qquad \frac{T}{2}+\frac{P_{ek}}{P_k} = \sqrt{\left(\frac{G}{P_k}(1-\mu^2)\right)}. \tag{6}$$

The sign depends on the magnitude of P_{ek}/P_k. In the following we take the upper sign if $P_{ek}/P_k \leq T/2$ and the lower sign if $P_{ek}/P_k > T/2$. Let

$$\Delta_1 = \sqrt{(1-\mu^2)}\pm\sqrt{(\lambda^2-\mu^2)}, \qquad \Delta = (1-\lambda^2)/\Delta_1 = \sqrt{(1-\mu^2)}\mp\sqrt{(\lambda^2-\mu^2)}.$$

After considerable algebra, in which we use the identity $4\mu^2+\Delta^2 = 2+2\lambda^2-\Delta_1^2$, we obtain

$$P_e = \frac{G}{4}(4\mu^2+\Delta^2), \qquad P_{ek} = \frac{G\Delta\Delta_1}{2T}, \qquad P_k = \frac{G\Delta_1^2}{T^2}. \tag{7}$$

The upper or lower sign is taken consistently in these expressions.

Now

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{P_k}}\int_{-T/2}^{T/2}\frac{\mathrm{d}z}{\sqrt{[(z+P_{ek}/P_k)^2+P_e/P_k-P_{ek}^2/P_k^2]}} \\ &= \frac{1}{\sqrt{P_k}}\log_e\left[\frac{P_{ek}+\frac{1}{2}TP_k+\sqrt{(P_eP_k+\frac{1}{4}T^2P_k^2+TP_{ek}P_k)}}{P_{ek}-\frac{1}{2}TP_k+\sqrt{(P_eP_k+\frac{1}{4}T^2P_k^2-TP_{ek}P_k)}}\right] \\ &= \frac{1}{\sqrt{P_k}}\left[\log_e\left(\frac{1+\sqrt{(1-\mu^2)}}{\mu}\right)\pm\log_e\left(\frac{\lambda+\sqrt{(\lambda^2-\mu^2)}}{\mu}\right)\right] = \frac{\psi}{\sqrt{P_k}} \quad \text{say.} \end{aligned} \tag{8}$$

$$I_2 = \frac{1}{\sqrt{P_k}}\int_{-T/2}^{T/2}\frac{z+P_{ek}/P_k}{y}\,\mathrm{d}z-\frac{P_{ek}}{P_k}I_1 \quad \text{where} \quad y = \sqrt{[(z+P_{ek}/P_k)^2+P_e/P_k-P_{ek}^2/P_k^2]}.$$

Defining $\varphi = \lambda-1$ and using (5), (6) and (7) gives

$$I_2 = -\frac{\varphi\sqrt{G}}{P_k}-\frac{T\Delta\psi}{2\Delta_1\sqrt{P_k}} \tag{9}$$

$$\begin{aligned} I_3 &= \frac{1}{\sqrt{P_k}}\int_{-T/2}^{T/2}\frac{z^2+2zP_{ek}/P_k+P_e/P_k}{y}\,\mathrm{d}z-2\frac{P_{ek}}{P_k}I_2-\frac{P_e}{P_k}I_1 \\ &= \frac{1}{\sqrt{P_k}}\left[\frac{1}{2}\left(z+\frac{P_{ek}}{P_k}\right)y\right]_{-T/2}^{T/2}+\frac{P_eP_k-P_{ek}^2}{2P_k^2}I_1+2\frac{P_{ek}^2}{P_k^2}I_1-\frac{P_e}{P_k}I_1+\frac{\varphi\sqrt{(G)}T\Delta}{P_k\Delta_1}. \end{aligned}$$

Depending on the magnitude of P_{ek}/P_k we get, again from (5)–(7)

$$I_3 = \frac{G}{2P_k^{\frac{3}{2}}}(\sqrt{(1-\mu^2)}\pm\lambda\sqrt{(\lambda^2-\mu^2)})+\frac{I_1}{2P_k}\mu^2G+\frac{T^2\Delta^2I_1}{2\Delta_1^2}-\frac{T^2}{4\Delta_1^2}(4\mu^2+\Delta^2)I_1+\frac{\varphi\sqrt{(G)}T\Delta}{P_k\Delta_1}.$$

Let $\chi = \sqrt{(1-\mu^2)}\pm\lambda\sqrt{(\lambda^2-\mu^2)}$, thus

$$I_3 = \frac{G\chi}{2P_k^{\frac{3}{2}}}+\frac{\psi T^2}{2\Delta_1^2\sqrt{P_k}}(\tfrac{1}{2}\Delta^2-\mu^2)+\frac{\varphi T^3\Delta}{\Delta_1^3\sqrt{G}}. \tag{10}$$

After further algebra, substitution of (7) to (10) in (1) finally gives

$$Q_t = \frac{1}{\Delta_1^2}(\mu^2\psi^2+\varphi^2)-P_q I_1^2/T^2 \tag{11}$$

$$Q_{tm} = -\frac{2}{\Delta_1^3}(\mu^2\Delta\psi^2+\Delta\varphi^2+\varphi\chi+\mu^2\varphi\psi)-4P_q I_1 I_2/T^3 \tag{12}$$

$$Q_m = \frac{4}{\Delta_1^4}[\mu^2\psi^2(\mu^2+\Delta^2)+(4\mu^2+\Delta^2)\varphi^2+2\mu^2\Delta\varphi\psi-2\mu^2\psi\chi+2\Delta\varphi\chi+\chi^2]-16P_q I_2^2/T^4 \tag{13}$$

$$Q_q = P_q I_1^2/T^2. \tag{14}$$

Apart from the P_q terms and the sign change for Q_{tm}, these are exactly the same as the expressions obtained by Ilyushin [2]. Note that if $I_2 = 0$ then Q_t+Q_q, Q_{tm} and Q_m remain unmodified by P_q. Denoting by a bar the expressions we would get for zero P_q, and letting $P_q = \beta P_e$, $(0 \le \beta \le 1)$, we substitute for I_1 and I_2 from (8) and (9):

$$Q_t = \bar{Q}_t-\frac{P_q\psi^2}{T^2 P_k} = \bar{Q}_t-\frac{\beta\psi^2}{4\Delta_1^2}(4\mu^2+\Delta^2) \tag{15}$$

hence

$$Q_q = \frac{\beta\psi^2}{4\Delta_1^2}(4\mu^2+\Delta^2) \tag{16}$$

$$Q_{tm} = \bar{Q}_{tm}-\frac{4\beta}{T^3}\frac{G}{4}(4\mu^2+\Delta^2)\frac{\psi}{\sqrt{P_k}}\left(-\frac{\varphi\sqrt{G}}{P_k}-\frac{T\Delta\psi}{2\Delta_1\sqrt{P_k}}\right)$$

$$= \bar{Q}_{tm}+\frac{\beta}{\Delta_1^3}(4\mu^2\varphi\psi+\Delta^2\varphi\psi+2\mu^2\Delta\psi^2+\tfrac{1}{2}\Delta^3\psi^2) \tag{17}$$

$$Q_m = \bar{Q}_m-\frac{16}{T^4}\beta\frac{G}{4}(4\mu^2+\Delta^2)\left(\varphi^2\frac{G}{P_k^2}+\frac{T\varphi\Delta\psi\sqrt{G}}{\Delta_1 P_k^{3/2}}+\frac{T^2\Delta^2\psi^2}{4\Delta_1^2 P_k}\right)$$

$$= \bar{Q}_m-4\frac{\beta}{\Delta_1^4}(4\mu^2\varphi^2+\Delta^2\varphi^2+4\mu^2\Delta\varphi\psi+\Delta^3\varphi\psi+\tfrac{1}{4}\Delta^4\psi^2+\mu^2\Delta^2\psi^2). \tag{18}$$

Having thus obtained the parametric representation of the yield surface we require to know the restrictions to be placed on λ, μ, β. It is evident from the definitions that $0 \le \lambda \ge \mu \le 1$. An application of the Schwarz inequality $|\mathbf{a}\,.\,\mathbf{b}| \le |\mathbf{a}|\,.\,|\mathbf{b}|$ where $\mathbf{a} = (e_1+\frac{1}{2}e_2, \frac{1}{2}\sqrt{(3)}e_2, \frac{1}{2}e_{12})$ and $\mathbf{b} = (k_1+\frac{1}{2}k_2, \frac{1}{2}\sqrt{(3)}k_2, \frac{1}{2}k_{12})$ shows that $P_{ek}^2 \le (P_e-P_q)P_k = (1-\beta)P_e P_k$.

$$\text{i.e.}\quad \frac{G^2\Delta^2\Delta_1^2}{4T^2} \le (1-\beta)\frac{G}{4}(4\mu^2+\Delta^2)\frac{G\Delta_1^2}{T^2} \quad\text{or}\quad \Delta^2 \le (1-\beta)(4\mu^2+\Delta^2)$$

$$\text{i.e.}\quad \beta \le \frac{4\mu^2}{4\mu^2+\Delta^2}. \tag{19}$$

4. ANALYSIS OF SOME APPROXIMATE YIELD SURFACES

Before performing a computer survey we will examine the boundary lines $\lambda = 1$ and $\mu = 0$. On $\lambda = 1$ we must have $P_{ek} = 0$ and hence $I_2 = 0$. Hence the resulting yield surface is exactly the same as given in Ref. [1] with Q_t replaced by Q_t+Q_q. On $\mu = 0$ (19) gives $\beta = 0$ and so $Q_q = 0$. Again therefore the yield surface is exactly as given in Ref. [1]. So we need to examine only $\mu > 0$ and $\lambda < 1$. The line $\lambda = \mu$ causes no convergence trouble except at (0, 0) or (1, 1) which need not be checked since these points lie on $\mu = 0$ or $\lambda = 1$.

In Ref. [1] it was suggested that Q_t be replaced by Q_t+Q_q. Accordingly we will make this substitution for the criteria Y_1, Y_3, Y_4 and the expression (2.3) of Ref. [1]. The modified Y_1 is thus $Q_t+Q_q+Q_m$ and it turns out that this function is simple to analyze. It is clear that increasing β does not affect Q_t+Q_q and tends to reduce Q_m, giving immediately (from Ref. [1])

$$Q_t+Q_q+Q_m \leq 1{\cdot}096. \tag{20}$$

We will obtain the lower limit by choosing the maximum β possible, i.e. $4\mu^2/(4\mu^2+\Delta^2)$.

Since we are considering $P_{ek} \geq 0$ only, it follows that $I_2 \leq 0$. Thus the extra terms in the modified Y_3, i.e. $Q_t+Q_q+Q_m+|Q_{tm}|/\sqrt{3}$ are

$$\frac{4P_q|I_2|}{T^3}\left[\frac{I_1}{\sqrt{3}}-\frac{4|I_2|}{T}\right]. \tag{21}$$

Similarly for Y_4 modified, which is $Q_t+Q_q+Q_m+2|Q_{tm}|$, we must add

$$\frac{8P_q|I_2|}{T^3}\left[I_1-2\frac{|I_2|}{T}\right]. \tag{22}$$

From the definitions of I_1 and I_2 we immediately have

$$|I_2| < TI_1/2. \tag{23}$$

Substituting (23) into (22) tells us that the modified Y_4 cannot be decreased by the addition of the shear terms and the maximum value for any λ and μ occurs with the highest β. Hence from Ref. [1] we get

$$1 \leq Q_t+Q_q+Q_m+2|Q_{tm}|. \tag{24}$$

No bounds can be placed on Y_3 but clearly the extreme limits must be attained either with $\beta = 0$ or β = maximum value = $4\mu^2/(4\mu^2+\Delta^2)$. Since $\beta = 0$ has been examined thoroughly in Ref. [1] it is necessary only to test $\beta = 4\mu^2/(4\mu^2+\Delta^2)$. The conclusion of all this is that only this value of β need be used in a computer survey to find the lower limit of Y'_1, the upper limit of Y'_4, and both the upper and lower limits of Y'_3, where $'$ denotes the modified function with Q_q added to Q_t. Since the function Y_2 of Ref. [1], i.e. $Q_t+\sqrt{Q_m}$, did not give bounds much better than Y_4, and was inferior to Y_3, it will not be considered here. It is in any case non-homogeneous and more difficult to analyze.

There remains the complicated expression (2.3) of Ref. [1] whose modification we will call Y'_6. Thus

$$Y'_6 = Q_t+Q_q+\tfrac{1}{2}Q_m-\frac{\frac{1}{4}[(Q_t+Q_q)Q_m-Q_{tm}^2]}{Q_t+Q_q+0{\cdot}48Q_m}+\sqrt{(\tfrac{1}{4}Q_m^2+Q_{tm}^2)}. \tag{25}$$

This function does not necessarily either increase or decrease with β for any given λ and μ. Since it was suggested [1] to use Y_6 as a checking function it is necessary to evaluate its new bounds when shear terms are included. Clearly this computation is the most time consuming since several values of β will have to be examined.

5. COMPUTATION AND RESULTS

The method of computation was essentially identical to that of Ref. [1]. First of all a survey was made on a (0·05 × 0·05) mesh of points in the $\lambda-\mu$ plane, omitting the lines $\mu = 0$ and $\lambda = 1$ where the behaviour of all the functions is known. This mesh size was sufficiently fine to ensure smooth and small (less than about 0·04) variation of all the yield functions between the points. After examining the results of this analysis, further computations were carried out on a very fine mesh in the neighbourhood of the absolute maximum or minimum of each yield function. The functions are well behaved and it is evident from the (0·05 × 0·05) results where the absolute maxima or minima must lie. Therefore we will not give a detailed discussion of the behaviour of the functions, as was done in Ref. [1]. All the results quoted are correct to three decimal places. In order to avoid confusion and to make comparison with the results of Ref. [1] possible, the modified and unmodified yield functions are defined again below. P_0 is the exact limit load for a theory which ignores transverse shear, and P_0' the limit load when transverse shear is taken into account.

$$Y_1 = Q_t + Q_m, \qquad Y_1' = Q_t + Q_q + Q_m$$

$$Y_3 = Q_t + Q_m + |Q_{tm}|/\sqrt{3}, \qquad Y_3' = Q_t + Q_q + Q_m + |Q_{tm}|/\sqrt{3}$$

$$Y_4 = Q_t + Q_m + 2|Q_{tm}|, \qquad Y_4' = Q_t + Q_q + Q_m + 2|Q_{tm}|$$

$$Y_6 = Q_t + \tfrac{1}{2}Q_m - \frac{\frac{1}{4}(Q_t Q_m - Q_{tm}^2)}{Q_t + 0{\cdot}48 Q_m} + \sqrt{(\tfrac{1}{4}Q_m^2 + Q_{tm}^2)}$$

$$Y_6' = Q_t + Q_q + \tfrac{1}{2}Q_m - \frac{\frac{1}{4}[(Q_t + Q_q)Q_m - Q_{tm}^2]}{Q_t + Q_q + 0{\cdot}48 Q_m} + \sqrt{(\tfrac{1}{4}Q_m^2 + Q_{tm}^2)}.$$

Let P_i denote the limit load for yield surface $Y_i = 1$ when shear is neglected, and P_i' the limit load for yield surface $Y_i' = 1$ when shear is allowed for. First of all it is clear that the unmodified yield functions Y_1 to Y_6 are hopelessly inaccurate when Q_q is the dominant term, and so no meaningful bounds exist for these functions relative to P_0'. In the limit, when $Q_q = 1$ and all the other terms are zero, they become infinitely unsafe. We have from Ref. [1],

$$0{\cdot}955 P_0 \le P_1 \le 1{\cdot}155 P_0 \tag{A1}$$

$$0{\cdot}939 P_0 \le P_3 \le 1{\cdot}034 P_0 \tag{A3}$$

$$0{\cdot}8 P_0 \le P_4 \le P_0 \tag{A4}$$

$$0{\cdot}999 P_0 \le P_6 \le 1{\cdot}005 P_0. \tag{A6}$$

Inequalities (A1) and (A4) have also been obtained by Gerdeen and Hutula [9].

Let us consider Y_1' first. It turns out that its minimum value occurs on $\mu = 0$ (where $Y_1' = Y_1$) and so its minimum is 0·75 (see Ref. [1]). It has been shown in inequality (20) that $Y_1' \le 1{\cdot}096$. Since all these approximate yield surfaces are homogeneous and second order in the stress resultants, and they have been evaluated for stress resultants lying *on* the exact yield surface, we must take the reciprocal of the square root of the maximum and minimum values to obtain the bounds on the load factor for the approximation

$Y'_i = 1$. Thus for Y'_1 we obtain the identical bounds as for (A1), i.e.

$$0{\cdot}955P'_0 \le P'_1 \le 1{\cdot}155P'_0. \tag{B1}$$

Now take Y'_3. Its maximum value is 1·16181 and its minimum occurs on $\mu = 0$, where it is 0·9347. This gives

$$0{\cdot}927P'_0 \le P'_3 \le 1{\cdot}034P'_0. \tag{B3}$$

Finally, the maximum value of Y'_4 is 1·59005, when $\mu = 0{\cdot}1$ approximately (not zero as it was for Y_4). Since we have already shown in (24) that $Y'_4 \ge 1$, we have

$$0{\cdot}793P'_0 \le P'_4 \le P'_0. \tag{B4}$$

When examining Y'_6 it unfortunately turns out to be almost invariably greater than unity, rising to about 1·11. Thus it is not a very good checking function in the general case, being inaccurate by as much as 5 per cent which is little better than Y'_3. It is therefore necessary to modify it still further, but the modification must be such that it vanishes when $\lambda = 1$ (i.e. $Q_{tm} = 0$) or when $\mu = 0$, i.e. $Q_q = 0$, since the approximation Y'_6 is very good in those cases. An obvious function which suggests itself is $r\sqrt{(Q_q|Q_{tm}|)}$ where r is an arbitrary constant. This function is subtracted from Y'_6. Several values of r were tried and it was found that $r = 0{\cdot}24$ gave the best results. The resulting yield function has a slightly better percentage error in its bounds than Y'_6, is more centrally balanced about the exact value, and reduces to Y'_6 when Y'_6 gives good results. Specifically we define

$$Y'_7 = Q_t + Q_q + \tfrac{1}{2}Q_m - \frac{\tfrac{1}{4}[(Q_t+Q_q)Q_m - Q_{tm}^2]}{Q_t+Q_q+0{\cdot}48Q_m} + \sqrt{(\tfrac{1}{4}Q_m^2 + Q_{tm}^2)} - 0{\cdot}24\sqrt{(Q_q|Q_{tm}|)}$$

and obtain

$$0{\cdot}978P'_0 \le P'_7 \le 1{\cdot}023P'_0. \tag{B6}$$

Possibly more refined functions could be constructed but it was felt that plus or minus 2·5 per cent error was quite accurate enough for practical purposes.

To conclude this section, mention must be made of a recent investigation of the effect of shear on the yield surface by Haydl and Sherbourne [10]. They use the yield surface

$$Q_t + Q_q + \tfrac{1}{2}Q_m + \sqrt{(\tfrac{1}{4}Q_m^2 + Q_{tm}^2)} = Y'_5 \quad \text{say.}$$

This is a modification of (2.2) of Ref. [1] and it is claimed that the bounds obtained in Ref. [1] need no modification if shear is included. Letting

$$Y_5 = Q_t + \tfrac{1}{2}Q_m + \sqrt{(\tfrac{1}{4}Q_m^2 + Q_{tm}^2)}$$

then (A5) of [1] is

$$0{\cdot}955P_0 \le P_5 \le P_0. \tag{A5}$$

Now if an analysis of Y'_5 is done in the same manner as for the other approximations it will be found that the correct bounds on P'_5 are

$$0{\cdot}938P'_0 \le P'_5 \le P'_0. \tag{B5}$$

Thus Haydl and Sherbourne's analysis, for this yield surface at least, appears to be in error. The most probable reason for this is that in [10] it is stated that Ilyushin's and Shapiro's yield surfaces are based on "average" yielding (through the thickness of the shell). This is a misunderstanding. Their parametric equations of the yield surface are derived on the assumption of yielding of every layer. They do however suggest *approximations* which may be interpreted as yielding on "average", i.e. Y_1 and Y'_1 of the present paper.

6. CONCLUSIONS

Various possible yield surfaces which allow for the effect of transverse shear have been examined and compared. It has been shown that if the yield functions Y_1, Y_3, Y_4 of Ref. [1] are modified by replacing Q_t by Q_t+Q_q, then the resulting bounds are very little different, if at all, from the bounds obtained in a theory where shear is neglected. Thus the suggestion of Ref. [1], that Q_t be replaced by Q_t+Q_q, has been found to be a good approximation. A new function has been suggested (Y'_7) which is accurate to within about plus or minus 2 per cent in all cases. It is recommended that a solution to a problem be obtained with either Y'_1 or Y'_3 (depending on how much computer time is available or how much accuracy desired) and that the final solution be checked with Y'_7. Y'_7 may also be used to examine solutions obtained by neglecting shear and so obtain an estimate of the error involved.

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REFERENCES

[1] M. ROBINSON, A comparison of yield surfaces for thin shells. *Int. J. Mech. Sci.* **13**, 345 (1971).
[2] A. A. ILYUSHIN, *Plasticity* (in Russian) Gostekhizdat, Moscow (1948); *Plasticité* (in French) Eyrolles, Paris (1956).
[3] Discussion of [1] and reply. *Int. J. Mech. Sci.* **14**, 531 (1972).
[4] A. SAWCZUK and M. DUSZEK, A note on the interaction of shear and bending in plastic plates. *Arch. Mech. St.* **15**, 411 (1963).
[5] F. ELLYIN and R. DELOIN, The effect of shear on yielding of structural members. *Int. J. Solids Struct.* **8**, 297 (1972).
[6] G. S. SHAPIRO, On yield surfaces for ideally plastic shells, *Problems of Continuum Mechanics* p. 414, Philadelphia (1961).
[7] J. HEYMAN, The full plastic moment of an I-beam in the presence of shear force. *J. Mech. Phys. Solids* **18**, 359 (1970).
[8] P. G. HODGE, JR., *Plastic Analysis of Structures.* McGraw-Hill (1959).
[9] J. C. GERDEEN and D. N. HUTULA, Plastic Collapse of ASME Ellipsoidal Head Pressure Vessels. *J. Eng. Indust.* **92**, 797 (1970).
[10] H. M. HAYDL and A. N. SHERBOURNE, Effect of transverse shear on limit load of cylindrical shells. *Nucl. Eng. Design* **22**, 290 (1972).

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Абстракт—Для общего случая, в котором поперечные напряжения сдвига не являются пренебрежимо малыми, исследуются некоторые приближенные поверхности течения для тонких оболочек. Находится, что если только сделать простое преобразование существующих приближений, тогда возникающие в результате пределы для предельной нагрузки почти тождественно равны такимже, полученными из элементарной теории оболочек, в которой пренебрегается поперечный сдвиг. Определяется, также, новое выражение, с точностью до плюс или минус 2 процентов.